

ALMOST AUTOMORPHIC DISTRIBUTIONS

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ABSTRACT. This work deals with almost automorphy of distributions. We give characterizations and main properties of these distributions. We also study the existence of distributional almost automorphic solutions of linear difference-differential equations.

1. INTRODUCTION

S. Bochner defined explicitly scalar-valued almost automorphic functions and gave some preliminary results on such functions in [1] and [2]. A general abstract study of almost automorphic functions is done in the work [6]. Since then almost automorphy becomes a subject of interest in mathematical research. It is well known that every Bohr almost periodic function is an almost automorphic one and that the space of almost automorphic functions is strictly larger than the space of almost periodic functions.

L. Schwartz introduced and studied almost periodic distributions in [4].

This work is aimed to introduce and to investigate Bochner almost automorphy in the setting of Schwartz-Sobolev distributions.

The paper is organised as follows, the second section recalls definitions and some properties of almost automorphic functions and the spaces \mathcal{D}_{L^p} with their topological duals. In the third one, we introduce smooth almost automorphic functions and give their main properties. Section four is devoted to the introduction of almost automorphic distributions and the study of their properties. Section five gives a characterization of almost automorphic distributions in the spirit of Bochner definition of almost automorphy of a scalar function. This characterisation will play an important role in applications. We show that Stepanov almost automorphic functions are examples of almost automorphic distributions. Finally in the last section as an application, we

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study the existence of distributional almost automorphic solutions of linear difference-differential equations.

2. PRELIMINARIES

In this work, we consider functions and distributions defined on the whole space \mathbb{R} . Let $(\mathcal{C}_b, \|\cdot\|_{L^\infty})$ denotes the space of continuous, bounded and complex valued functions on \mathbb{R} endowed with the norm of uniform convergence on \mathbb{R} , it is well-known that $(\mathcal{C}_b, \|\cdot\|_{L^\infty})$ is a Banach algebra.

Definition 1. (*S. Bochner [1]*) A complex-valued function f defined and continuous on \mathbb{R} is called almost automorphic, if for any sequence $(s_m)_{m \in \mathbb{N}} \subset \mathbb{R}$, one can extract a subsequence $(s_{m_k})_k$ such that

$$(2.1) \quad g(x) := \lim_{k \rightarrow +\infty} f(x + s_{m_k}) \text{ is well-defined for every } x \in \mathbb{R},$$

and

$$(2.2) \quad \lim_{k \rightarrow +\infty} g(x - s_{m_k}) = f(x) \text{ for every } x \in \mathbb{R}.$$

The space of almost automorphic functions on \mathbb{R} is denoted by \mathcal{C}_{aa} .

Remark 1. The function g in the above definition is not necessary continuous but it is measurable and bounded, so locally integrable.

Remark 2. We have the strict inclusion $\mathcal{C}_{ap} \subsetneq \mathcal{C}_{aa}$, where \mathcal{C}_{ap} is the space of Bohr almost periodic functions, see [6].

The following characterization of almost automorphic functions is given in [2].

Proposition 1. A complex-valued function f defined and continuous on \mathbb{R} is almost automorphic if and only if for any sequence of real numbers $(s_m)_{m \in \mathbb{N}}$, one can extract a subsequence $(s_{m_k})_k$ such that

$$\lim_{l \rightarrow +\infty} \lim_{k \rightarrow +\infty} f(x + s_{m_k} - s_{m_l}) = f(x), \quad \forall x \in \mathbb{R}.$$

Some properties of almost automorphic functions are summarized in the following proposition.

Proposition 2. (1) \mathcal{C}_{aa} is a Banach subalgebra of \mathcal{C}_b .

(2) Let $f \in \mathcal{C}_{aa}$ and $\lim_{x \rightarrow \infty} f(x) = 0$, then $f = 0$.

(3) Let $f \in \mathcal{C}_{aa}$ and $h \in \mathbb{R}$, then $\tau_h f := f(\cdot + h) \in \mathcal{C}_{aa}$.

(4) If $f \in \mathcal{C}_{aa}$ and $g \in L^1$, then the convolution $f * g \in \mathcal{C}_{aa}$.

The condition of almost automorphy of a primitive has been established in [8].

Proposition 3. *A primitive of an almost automorphic function is almost automorphic if and only if it is bounded.*

For more details on the following part see [4] and [5]. Let \mathcal{C}^∞ be the space of infinitely differentiable functions on \mathbb{R} and $p \in [1, +\infty]$, the space

$$\mathcal{D}_{L^p} := \{\varphi \in \mathcal{C}^\infty : \forall i \in \mathbb{Z}_+, \varphi^{(i)} \in L^p\}$$

endowed with the topology defined by the countable family of norms

$$|\varphi|_{m,p} = \sum_{i=0}^m \|\varphi^{(i)}\|_{L^p}, \quad m \in \mathbb{Z}_+,$$

is a Fréchet subalgebra of \mathcal{C}^∞ .

The space $\dot{\mathcal{B}}$ is by definition the closure in \mathcal{D}_{L^∞} of the space \mathcal{D} of smooth functions with compact support, it is the space of functions from \mathcal{D}_{L^∞} which vanish at infinity with all their derivatives. It is clear that the space \mathcal{D} is dense in \mathcal{D}_{L^p} , $1 \leq p < +\infty$, and $\dot{\mathcal{B}}$.

The space of L^p -distributions, denoted by \mathcal{D}'_{L^p} , $1 < p \leq +\infty$, is the topological dual space of \mathcal{D}_{L^q} , where $\frac{1}{p} + \frac{1}{q} = 1$. The topological dual of $\dot{\mathcal{B}}$ is denoted by \mathcal{D}'_{L^1} . We have the following characterizations of \mathcal{D}'_{L^p} , $1 \leq p \leq +\infty$.

Proposition 4. *Let $T \in \mathcal{D}'$, the following statements are equivalent :*

- (1) $T \in \mathcal{D}'_{L^p}$.
- (2) $T * \varphi \in L^p, \forall \varphi \in \mathcal{D}$.
- (3) $\exists (f_j)_{j \leq k} \subset L^p, T = \sum_{j=0}^k f_j^{(j)}$.

Remark 3. *A distribution from the space \mathcal{D}'_{L^∞} is called a bounded distribution.*

3. SMOOTH ALMOST AUTOMORPHIC FUNCTIONS

The space of smooth almost automorphic functions on \mathbb{R} is denoted and defined as follows

$$\mathcal{B}_{aa} := \{\varphi \in \mathcal{C}^\infty : \forall i \in \mathbb{Z}_+, \varphi^{(i)} \in \mathcal{C}_{aa}\}.$$

We note that $\mathcal{B}_{aa} \subset \mathcal{D}_{L^\infty}$ and that \mathcal{B}_{aa} is endowed with the induced topology of \mathcal{D}_{L^∞} .

Here are some properties of the space \mathcal{B}_{aa} .

- Proposition 5.**
- (1) $\mathcal{B}_{aa} = \mathcal{C}_{aa} \cap \mathcal{D}_{L^\infty}$.
 - (2) \mathcal{B}_{aa} is a Fréchet subalgebra of \mathcal{D}_{L^∞} .
 - (3) $\mathcal{B}_{aa} * L^1 \subset \mathcal{B}_{aa}$.

Proof. 1. It is clear that $\mathcal{B}_{aa} \subset \mathcal{C}_{aa} \cap \mathcal{D}_{L^\infty}$. Now let $f \in \mathcal{C}_{aa} \cap \mathcal{D}_{L^\infty}$, then by the mean value theorem, we get

$$|f^{(i)}(x) - f^{(i)}(y)| \leq \sup_{z \in \mathbb{R}} |f^{(i+1)}(z)| |x - y|, \quad \forall i \in \mathbb{Z}_+, \quad \forall x, y \in \mathbb{R},$$

this gives that $\forall i \in \mathbb{Z}_+$, $f^{(i)}$ is uniformly continuous, and it is known that the derivative of almost automorphic function is also almost automorphic if and only if it is uniformly continuous [6], so $\forall i \in \mathbb{Z}_+$, $f^{(i)} \in \mathcal{C}_{aa}$, i.e. $f \in \mathcal{B}_{aa}$.

2. As the topology of $\mathcal{B}_{aa} \subset \mathcal{D}_{L^\infty}$ is given by the countable family of submultiplicative norms $|\cdot|_{k,\infty}$, $k \in \mathbb{Z}_+$, it remains to show the completeness of \mathcal{B}_{aa} . Let $(f_m)_{m \in \mathbb{N}} \subset \mathcal{B}_{aa}$ be a Cauchy sequence, it is clear that $\forall i \in \mathbb{Z}_+$, $(f_m^{(i)})_{m \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{C}_{aa} , and since $(\mathcal{C}_{aa}, \|\cdot\|_{L^\infty})$ is complete, then $\forall i \in \mathbb{Z}_+$, $f_m^{(i)}$ converges uniformly to $f_i \in \mathcal{C}_{aa}$, setting $f_0 = f$, we obtain, due to the uniform convergence, that $f \in \mathcal{C}^\infty$ and $\forall i \in \mathbb{Z}_+$, $f^{(i)} = f_i \in \mathcal{C}_{aa}$, i.e. $(f_m)_{m \in \mathbb{N}}$ converges to f in the topology of \mathcal{B}_{aa} , which means that \mathcal{B}_{aa} is complete.

3. If $h \in L^1$ and $f \in \mathcal{B}_{aa}$, then $(f * h) \in \mathcal{C}^\infty$ and $\forall i \in \mathbb{Z}_+$, $(f * h)^{(i)} = f^{(i)} * h \in \mathcal{C}_{aa}$ by proposition 2 – (3). \square

Remark 4. We have $\mathcal{B}_{aa} \subsetneq \mathcal{C}^\infty \cap \mathcal{C}_{aa}$.

The following result is a consequence of the above proposition.

Corollary 1. Let $f \in \mathcal{D}_{L^\infty}$, then $f \in \mathcal{B}_{aa}$ if and only if $\forall \varphi \in \mathcal{D}$, $f * \varphi \in \mathcal{C}_{aa}$.

Proof. The necessity is clear from the above proposition. Let $f \in \mathcal{D}_{L^\infty}$, take a sequence $(\rho_m)_{m \in \mathbb{N}} \subset \mathcal{D}$ such that $\rho_m \geq 0$, $\text{supp} \rho_m \subset [-\frac{1}{m}, \frac{1}{m}]$ and $\int_{\mathbb{R}} \rho_m(x) dx = 1$, by hypothesis $f * \rho_m \in \mathcal{C}_{aa}$, $\forall m \in \mathbb{N}$. As

$$\begin{aligned} |f * \rho_m(x) - f(x)| &= \left| \int_{\mathbb{R}} \rho_m(y) (f(x-y) - f(x)) dy \right|, \\ &\leq \sup_{y \in \mathbb{R}} |f'(y)| \int_{-\frac{1}{m}}^{\frac{1}{m}} \rho_m(y) |y| dy, \\ &\leq \frac{1}{m} \sup_{z \in \mathbb{R}} |f'(z)|, \end{aligned}$$

so $(f * \rho_m)_{m \in \mathbb{N}}$ converges uniformly to f on \mathbb{R} , by proposition 2 – (1), $f \in \mathcal{C}_{aa}$, i.e. $f \in \mathcal{D}_{L^\infty} \cap \mathcal{C}_{aa} = \mathcal{B}_{aa}$. \square

4. ALMOST AUTOMORPHIC DISTRIBUTIONS

The goal of this section is the introduction of almost automorphic distributions and the study of some of their properties.

Theorem 1. *Let $T \in \mathcal{D}'_{L^\infty}$, the following statements are equivalent :*

- (1) $T * \varphi \in \mathcal{C}_{aa}, \forall \varphi \in \mathcal{D}.$
- (2) $\exists (f_j)_{j \leq k} \subset \mathcal{C}_{aa}, T = \sum_{j=0}^k f_j^{(j)}.$

Proof. $1 \Rightarrow 2.$ Let $T \in \mathcal{D}'_{L^\infty}$, then $\exists m \in \mathbb{Z}_+, \exists C > 0$ such that

$$|\langle T, \psi \rangle| \leq C |\psi|_{m,1}, \forall \psi \in \mathcal{D}_{L^1}.$$

For $p \in \mathbb{N}$, we consider a fundamental solution G of the differential operator $\left(1 - \frac{d^2}{dx^2}\right)^p$ which satisfies $G \in \mathcal{C}^{2p-2}$ and with integrable derivatives of order $\leq 2p-2$, see [4]. As $G \in L^1$ and $T \in \mathcal{D}'_{L^\infty}$, then $T * G$ exists and we have

$$T = \left(1 - \frac{d^2}{dx^2}\right)^p (G * T).$$

So $G \in \mathcal{D}_{L^1}^{2m+2} := \{\varphi \in \mathcal{C}^{2m+2} : \forall j \leq 2m+2, \varphi^{(j)} \in L^1\}$, if $p = m+2$. The space $\mathcal{D}_{L^1}^{2m+2}$ is endowed with the norm $|\cdot|_{2m+2,1}$. It is well known that \mathcal{D} is dense in $\mathcal{D}_{L^1}^{2m+2}$, as $\mathcal{D} \subset \mathcal{D}_{L^1} \subset \mathcal{D}_{L^1}^{2m+2} \subset \mathcal{D}_{L^1}^m$, then T is extended continuously to the space $\mathcal{D}_{L^1}^{2m+2}$ and there exists a sequence $(\theta_k)_{k \in \mathbb{N}} \subset \mathcal{D}$ such that $(\theta_k)_k$ converges to G in $\mathcal{D}_{L^1}^{2m+2}$. We also have

$$\begin{aligned} |(T * \theta_k)(x) - (T * G)(x)| &= |\langle T, \tau_{-x} \check{\theta}_k - \tau_{-x} \check{G} \rangle|, \\ &\leq C |\theta_k - G|_{2m+2,1}, \end{aligned}$$

which gives

$$\sup_{x \in \mathbb{R}} |(T * \theta_k)(x) - (T * G)(x)| \xrightarrow[k \rightarrow +\infty]{} 0,$$

i.e. the sequence $(T * \theta_k)_{k \in \mathbb{N}}$ converges uniformly to $T * G$ on \mathbb{R} , by proposition 2 – (1) we get $T * G \in \mathcal{C}_{aa}$.

$2 \Rightarrow 1.$ For $\varphi \in \mathcal{D}, T * \varphi = \sum_{j=0}^k f_j^{(j)} * \varphi = \sum_{j=0}^k f_j * \varphi^{(j)} \in \mathcal{C}_{aa}$ due to proposition 2 – (3). \square

Definition 2. *A distribution $T \in \mathcal{D}'_{L^\infty}$ is said almost automorphic if it satisfies any (hence every) condition of the above theorem. We denote by \mathcal{B}'_{aa} the space of all almost automorphic distributions on \mathbb{R} .*

Remark 5. *The theorem says that $T \in \mathcal{D}'_{L^\infty}$ is almost automorphic if and only if there exist a function $f \in \mathcal{C}_{aa}$ and $m \in \mathbb{Z}_+$ such that $T = \left(1 - \frac{d^2}{dx^2}\right)^{m+2} f$, where m is the order of T .*

As a consequence of the theorem, we obtain the following result.

Corollary 2. *Let $(T_m)_{m \in \mathbb{N}} \subset \mathcal{B}'_{aa}$ and $T \in \mathcal{D}'_{L^\infty}$ such that $\forall \varphi \in \mathcal{D}, (T_m * \varphi)_{m \in \mathbb{N}}$ converges uniformly to $T * \varphi$, then $T \in \mathcal{B}'_{aa}$.*

Example 1. *Every classical almost automorphic function is an almost automorphic distribution.*

Example 2. *Let \mathcal{B}'_{ap} , see [4], be the space of almost periodic Schwartz distributions, then we have the strict inclusion $\mathcal{B}'_{ap} \subsetneq \mathcal{B}'_{aa}$.*

The translate $\tau_h T$, $h \in \mathbb{R}$, of a distribution $T \in \mathcal{D}'$ is defined by

$$\langle \tau_h T, \varphi \rangle = \langle T, \tau_{-h} \varphi \rangle, \quad \forall \varphi \in \mathcal{D},$$

where

$$\tau_{-h} \varphi(x) = \varphi(x - h), \quad \forall x \in \mathbb{R}.$$

Denote by $\mathcal{B}'_{+,0}$ the space of $T \in \mathcal{D}'$ such that $\lim_{h \rightarrow +\infty} \tau_h T = 0$ in \mathcal{D}' .

The following proposition summarises the main properties of \mathcal{B}'_{aa} .

Proposition 6. (1) *If $T \in \mathcal{B}'_{aa}$, then $T^{(i)} \in \mathcal{B}'_{aa}$, $\forall i \in \mathbb{Z}_+$.*

(2) *If $T \in \mathcal{B}'_{aa} \cap \mathcal{B}'_{+,0}$, then $T = 0$.*

(3) *If $T \in \mathcal{B}'_{aa}$, then $\tau_h T \in \mathcal{B}'_{aa}$, $\forall h \in \mathbb{R}$.*

(4) *$\mathcal{B}'_{aa} * \mathcal{D}'_{L^1} \subset \mathcal{B}'_{aa}$.*

(5) *$\mathcal{B}'_{aa} \times \mathcal{B}'_{aa} \subset \mathcal{B}'_{aa}$.*

Proof. 1. Obvious.

2. As $T \in \mathcal{B}'_{+,0}$, then $\forall \varphi \in \mathcal{D}, \lim_{x \rightarrow +\infty} (T * \varphi)(x) = \lim_{x \rightarrow +\infty} \langle T, \tau_{-x} \check{\varphi} \rangle = 0$ and as $T \in \mathcal{B}'_{aa}$, $T * \varphi \in \mathcal{C}_{aa}$, by proposition 2 – (2), we get $T * \varphi \equiv 0$, $\forall \varphi \in \mathcal{D}$. On the other hand $\langle T, \varphi \rangle = (T * \check{\varphi})(0) = 0$, so $T = 0$.

3. Let $T \in \mathcal{B}'_{aa}$, then $\forall \varphi \in \mathcal{D}$ and $\forall h \in \mathbb{R}, \tau_h T * \varphi = \tau_h (T * \varphi)$, as $T * \varphi \in \mathcal{C}_{aa}$ and \mathcal{C}_{aa} is invariant by translation, then $\tau_h (T * \varphi) \in \mathcal{C}_{aa}$, so $\tau_h T \in \mathcal{B}'_{aa}$.

4. Let $T \in \mathcal{B}'_{aa}$ and $S \in \mathcal{D}'_{L^1}$, there exist $(f_j)_{j \leq k} \subset \mathcal{C}_{aa}$ such that $T = \sum_{j=0}^k f_j^{(j)}$ and $(g_j)_{j \leq m} \subset L^1$ such that $S = \sum_{j=0}^m g_j^{(j)}$, these give

$$(T * S) = \sum_{l=0}^k \sum_{j=0}^m (f_l * g_j)^{(l+j)},$$

by proposition 2 – (3), $f_i * g_j \in \mathcal{C}_{aa}$, since $\mathcal{D}'_{L^\infty} * \mathcal{D}'_{L^1} \subset \mathcal{D}'_{L^\infty}$, we have $T * S \in \mathcal{B}'_{aa}$.

5. Let $T \in \mathcal{B}'_{aa}$, there exists $(f_j)_{j \leq k} \subset \mathcal{C}_{aa}$, such that $T = \sum_{j=0}^k f_j^{(j)}$.

For $\varphi \in \mathcal{B}_{aa}$, we have

$$\varphi T = \sum_{j=0}^k \varphi f_j^{(j)} = \sum_{j=0}^k \sum_{l=0}^j (-1)^l \binom{j}{l} (\varphi^{(l)} f_j)^{(j-l)},$$

and since \mathcal{C}_{aa} is an algebra, then $\varphi^{(j)} f_i \in \mathcal{C}_{aa}$, hence $\varphi T \in \mathcal{B}'_{aa}$. \square

The next result shows that \mathcal{B}_{aa} is dense in \mathcal{B}'_{aa} .

Proposition 7. *A distribution $T \in \mathcal{D}'_{L^\infty}$ is almost automorphic if and only if there exists $(\varphi_m)_{m \in \mathbb{N}} \subset \mathcal{B}_{aa}$ such that $\lim_{m \rightarrow +\infty} \varphi_m = T$ in \mathcal{D}'_{L^∞} .*

Proof. Suppose that there exists $(\varphi_m)_{m \in \mathbb{N}} \subset \mathcal{B}_{aa}$ such that $\lim_{m \rightarrow +\infty} \varphi_m = T$ in \mathcal{D}'_{L^∞} . By the strong topology of \mathcal{D}'_{L^∞} , for any bounded subset $B \subset \mathcal{D}_{L^1}$ we have

$$\sup_{\psi \in B} |\langle \varphi_m - T, \psi \rangle| \xrightarrow{m \rightarrow +\infty} 0.$$

For a fixed $\varphi \in \mathcal{D}$, the set $B := \{\tau_{-x}\check{\varphi} : x \in \mathbb{R}\}$ is bounded in \mathcal{D}_{L^1} , so

$$\begin{aligned} \sup_{x \in \mathbb{R}} |(\varphi_m * \varphi)(x) - (T * \varphi)(x)| &= \sup_{x \in \mathbb{R}} |\langle \varphi_m - T, \tau_{-x}\check{\varphi} \rangle| \\ &= \sup_{\psi \in B} |\langle \varphi_m - T, \psi \rangle| \xrightarrow{m \rightarrow +\infty} 0, \end{aligned}$$

i.e. the sequence of functions $(\varphi_m * \varphi)_{m \in \mathbb{N}} \subset \mathcal{C}_{aa}$ converges uniformly to $(T * \varphi)$, by propositions 2 – (1), $T * \varphi \in \mathcal{C}_{aa}$, $\forall \varphi \in \mathcal{D}$, so $T \in \mathcal{B}'_{aa}$.

Conversely, let $T \in \mathcal{B}'_{aa}$ and take a sequence of positive test functions $(\rho_m)_{m \in \mathbb{N}} \subset \mathcal{D}$ such that $\text{supp} \rho_m \subset [-\frac{1}{m}, \frac{1}{m}]$ and $\int_{\mathbb{R}} \rho_m(x) dx = 1$.

1. Define $\varphi_m := \rho_m * T \in \mathcal{B}_{aa}$. We claim that for any bounded set U of \mathcal{D}_{L^1} , $\sup_{\varphi \in U} |\langle \varphi_m - T, \varphi \rangle| \xrightarrow{m \rightarrow +\infty} 0$. Indeed, since $T \in \mathcal{D}'_{L^\infty}$, $\exists l \in \mathbb{Z}_+$, $\exists C > 0$, $|\langle T, \varphi \rangle| \leq C |\varphi|_{l,1}$, $\forall \varphi \in \mathcal{D}_{L^1}$, and then

$$|\langle \varphi_m - T, \varphi \rangle| = |\langle T, \check{\rho}_m * \varphi - \varphi \rangle| \leq C |\check{\rho}_m * \varphi - \varphi|_{l,1}, \quad \forall \varphi \in \mathcal{D}_{L^1}.$$

On the other hand,

$$(\check{\rho}_m * \varphi)^{(i)}(x) - \varphi^{(i)}(x) = \int_{\mathbb{R}} \check{\rho}_m(y) (\varphi^{(i)}(x-y) - \varphi^{(i)}(x)) dy,$$

by Minkowski inequality and the mean value theorem we obtain for a $t \in]0, 1[$,

$$\begin{aligned}
\left\| (\check{\rho}_m * \varphi)^{(i)} - \varphi^{(i)} \right\|_{L^1} &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \check{\rho}_m(y) (\varphi^{(i)}(x-y) - \varphi^{(i)}(x)) dy \right| dx \\
&\leq \int_{\left[\frac{-1}{m}, \frac{1}{m}\right]} \check{\rho}_m(y) \int_{\mathbb{R}} |y| |\varphi^{(i+1)}(x + (t-1)y)| dx dy, \\
&\leq \int_{\left[\frac{-1}{m}, \frac{1}{m}\right]} |y| \check{\rho}_m(y) \int_{\mathbb{R}} |\varphi^{(i+1)}(z)| dz dy, \\
&\leq \frac{1}{m} \|\varphi^{(i+1)}\|_{L^1},
\end{aligned}$$

hence

$$|\langle \varphi_m - T, \varphi \rangle| \leq C \|\check{\rho}_m * \varphi - \varphi\|_{l,1} \leq \frac{C}{m} \|\varphi\|_{l+1,1}, \forall \varphi \in \mathcal{D}_{L^1}.$$

Take U a bounded set in \mathcal{D}_{L^1} , then $\exists M, \forall \varphi \in U, \|\varphi\|_{l+1,1} \leq M$, consequently we obtain

$$\sup_{\varphi \in U} |\langle \varphi_m - T, \varphi \rangle| \leq \frac{MC}{m} \xrightarrow{m \rightarrow +\infty} 0,$$

which gives the conclusion. \square

The following result is an extension to almost automorphic distributions of the classical result of Bohl-Bohr on primitives.

Proposition 8. *A primitive of an almost automorphic distribution is almost automorphic if and only if it is bounded.*

Proof. If $T \in \mathcal{B}'_{aa}$ and its primitive $S \in \mathcal{B}'_{aa}$, it is clear that $S \in \mathcal{D}'_{L^\infty}$. Conversely, let $S \in \mathcal{D}'_{L^\infty}$ be a primitive of $T \in \mathcal{B}'_{aa}$, i.e. $S' = T$, so $S * \varphi \in L^\infty, \forall \varphi \in \mathcal{D}$, and

$$(S * \varphi)' = S' * \varphi = T * \varphi \in \mathcal{C}_{aa}, \forall \varphi \in \mathcal{D},$$

i.e. $(S * \varphi)$ is a bounded primitive of the almost automorphic function $(T * \varphi)$, thus by proposition 3, $S * \varphi \in \mathcal{C}_{aa}, \forall \varphi \in \mathcal{D}$, so $S \in \mathcal{B}'_{aa}$. \square

5. BOCHNER ALMOST AUTOMORPHY OF DISTRIBUTIONS

This section gives a characterization of almost automorphic distributions in the spirit of the topological definition of almost automorphy

of a scalar function given by S. Bochner. We also show that almost automorphic functions in the sense of Stepanov are almost automorphic distributions.

Theorem 2. *Let $T \in \mathcal{D}'_{L^\infty}$, the following propositions are equivalent :*

- (1) *There exists $(f_j)_{j \leq n} \subset \mathcal{C}_{aa}$ such that $T = \sum_{j=0}^n f_j^{(j)}$.*
- (2) *For every sequence $(s_m)_{m \in \mathbb{N}} \subset \mathbb{R}$, there exists a subsequence $(s_{m_k})_k$ such that*

$$S := \lim_{k \rightarrow +\infty} \tau_{s_{m_k}} T \text{ exists in } \mathcal{D}',$$

and

$$\lim_{l \rightarrow +\infty} \tau_{-s_{m_l}} S = T \text{ in } \mathcal{D}'.$$

- (3) $T * \varphi \in \mathcal{C}_{aa}, \forall \varphi \in \mathcal{D}$.

Proof. $1 \Rightarrow 2$. Let $(f_j)_{j \leq n} \subset \mathcal{C}_{aa}$ and $T = \sum_{j=0}^n f_j^{(j)}$, for every sequence

$(s_m)_{m \in \mathbb{N}} \subset \mathbb{R}$, there is a subsequence $(s_{m_{k_1}})_{k_1}$ such that $\forall x \in \mathbb{R}$,

$$\lim_{k_1 \rightarrow +\infty} f_1(x + s_{m_{k_1}}) = g_1(x) \text{ and } \lim_{k_1 \rightarrow +\infty} g_1(x - s_{m_{k_1}}) = f_1(x),$$

since f_2 is also an almost automorphic, one extracts a subsequence $(s_{m_{k_2}})_{k_2}$ of $(s_{m_{k_1}})_{k_1}$ such that $\forall x \in \mathbb{R}$,

$$\lim_{k_2 \rightarrow +\infty} f_2(x + s_{m_{k_2}}) = g_2(x) \text{ and } \lim_{k_2 \rightarrow +\infty} g_2(x - s_{m_{k_2}}) = f_2(x),$$

and also,

$$\lim_{k_2 \rightarrow +\infty} f_1(x + s_{m_{k_2}}) = g_1(x) \text{ and } \lim_{k_2 \rightarrow +\infty} g_1(x - s_{m_{k_2}}) = f_1(x),$$

hence by iterating this process, we can extract a subsequence $(s_{m_{k_n}})_{k_n}$ of $(s_{m_{k_{n-1}}})_{k_{n-1}}$, which is also a subsequence of $(s_m)_{m \in \mathbb{N}}$ such that $\forall x \in \mathbb{R}$,

$$\lim_{k_n \rightarrow +\infty} f_i(x + s_{m_{k_n}}) = g_i(x) \text{ and } \lim_{k_n \rightarrow +\infty} g_i(x - s_{m_{k_n}}) = f_i(x), \quad i = 1, \dots, n.$$

On the other hand, the function f_i is bounded and $g_i \in L^1_{loc}$, so $\forall \varphi \in \mathcal{D}$,

$$\langle \tau_{s_{m_{k_n}}} T, \varphi \rangle = \sum_{i=0}^n (-1)^i \int_{\mathbb{R}} f_i(x + s_{m_{k_n}}) \varphi^{(i)}(x) dx,$$

then by Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{k_n \rightarrow +\infty} \left\langle \tau_{s_{m_{k_n}}} T, \varphi \right\rangle &= \sum_{i=0}^n (-1)^i \int_{\mathbb{R}} g_i(x) \varphi^{(i)}(x) dx, \\ &= \left\langle \sum_{i=0}^n g_i^{(i)}, \varphi \right\rangle, \\ &= \langle S, \varphi \rangle, \end{aligned}$$

where $S := \sum_{i=0}^n g_i^{(i)} \in \mathcal{D}'$. In the same way, since the function g_i is measurable and bounded, we obtain $\forall \varphi \in \mathcal{D}$,

$$\begin{aligned} \lim_{k_n \rightarrow +\infty} \left\langle \tau_{-s_{m_{k_n}}} S, \varphi \right\rangle &= \sum_{i=0}^n (-1)^i \int_{\mathbb{R}} \lim_{k_n \rightarrow +\infty} g_i(x - s_{m_{k_n}}) \varphi^{(i)}(x) dx, \\ &= \left\langle \sum_{i=0}^n f_i^{(i)}, \varphi \right\rangle, \\ &= \langle T, \varphi \rangle, \end{aligned}$$

i.e.

$$\lim_{k_n \rightarrow +\infty} \tau_{-s_{m_{k_n}}} S = T \text{ in } \mathcal{D}'.$$

2 \Rightarrow 3. Let $\varphi \in \mathcal{D}$, for any sequence $(s_m)_{m \in \mathbb{N}} \subset \mathbb{R}$, one can extract a subsequence $(s_{m_k})_{k \in \mathbb{N}}$ such that $\forall x \in \mathbb{R}$, we have

$$\begin{aligned} \lim_{k \rightarrow +\infty} (T * \varphi)(x + s_{m_k}) &= \lim_{k \rightarrow +\infty} \langle \tau_{s_{m_k}} T, \tau_{-x} \check{\varphi} \rangle, \\ &= \langle S, \tau_{-x} \check{\varphi} \rangle, \\ &= (S * \varphi)(x), \end{aligned}$$

and in the same way $\forall x \in \mathbb{R}$,

$$\begin{aligned} \lim_{k \rightarrow +\infty} (S * \varphi)(x - s_{m_k}) &= \lim_{k \rightarrow +\infty} \langle \tau_{-s_{m_k}} S, \tau_{-x} \check{\varphi} \rangle, \\ &= \langle T, \tau_{-x} \check{\varphi} \rangle, \\ &= (T * \varphi)(x), \end{aligned}$$

i.e. $T * \varphi \in \mathcal{C}_{aa}$, $\forall \varphi \in \mathcal{D}$.

3 \Rightarrow 1. Follows from theorem 1. \square

The equivalent properties established in the last theorem lead to a Bochner type definition of an almost automorphic distribution, i.e. a

distribution $T \in \mathcal{D}'_{L^\infty}$ is an almost automorphic distribution if for every sequence $(s_m)_{m \in \mathbb{N}} \subset \mathbb{R}$, there is a subsequence $(s_{m_k})_k$ such that

$$S := \lim_{k \rightarrow +\infty} \tau_{s_{m_k}} T \text{ exists in } \mathcal{D}',$$

and

$$\lim_{k \rightarrow +\infty} \tau_{-s_{m_k}} S = T \text{ in } \mathcal{D}'.$$

We have also an extension of the proposition 1 to distributions.

Proposition 9. *Let $T \in \mathcal{D}'_{L^\infty}$, then $T \in \mathcal{B}'_{aa}$ if and only if for every sequence $(s_m)_{m \in \mathbb{N}} \subset \mathbb{R}$, there exists a subsequence $(s_{m_k})_k$ such that*

$$(5.1) \quad \lim_{l \rightarrow +\infty} \lim_{k \rightarrow +\infty} \tau_{-s_{m_l}} \tau_{s_{m_k}} T = T \text{ in } \mathcal{D}'.$$

Proof. If $T \in \mathcal{B}'_{aa}$, by the above theorem, for every sequence $(s_m)_{m \in \mathbb{N}} \subset \mathbb{R}$, there is a subsequence $(s_{m_k})_k$ such that

$$S := \lim_{k \rightarrow +\infty} \tau_{s_{m_k}} T \text{ and } \lim_{k \rightarrow +\infty} \tau_{-s_{m_k}} S = T \text{ hold in } \mathcal{D}'.$$

Let $\varphi \in \mathcal{D}$,

$$\begin{aligned} \lim_{k \rightarrow +\infty} \langle \tau_{-s_{m_l}} \tau_{s_{m_k}} T, \varphi \rangle &= \lim_{k \rightarrow +\infty} \langle \tau_{s_{m_k}} T, \tau_{s_{m_l}} \varphi \rangle, \\ &= \langle S, \tau_{s_{m_l}} \varphi \rangle, \end{aligned}$$

and

$$\begin{aligned} \lim_{l \rightarrow +\infty} \lim_{k \rightarrow +\infty} \langle \tau_{-s_{m_l}} \tau_{s_{m_k}} T, \varphi \rangle &= \lim_{l \rightarrow +\infty} \langle \tau_{-s_{m_l}} S, \varphi \rangle, \\ &= \langle T, \varphi \rangle, \end{aligned}$$

i.e. $\lim_{l \rightarrow +\infty} \lim_{k \rightarrow +\infty} \tau_{-s_{m_l}} \tau_{s_{m_k}} T = T$ in \mathcal{D}' .

Conversely, let a sequence $(s_m)_{m \in \mathbb{N}} \subset \mathbb{R}$ with a subsequence $(s_{m_k})_k$ such that (5.1) holds, then $\forall \varphi \in \mathcal{D}$ and $\forall x \in \mathbb{R}$, we have

$$\begin{aligned} \lim_{l \rightarrow +\infty} \lim_{k \rightarrow +\infty} (T * \varphi)(x + s_{m_k} - s_{m_l}) &= \lim_{l \rightarrow +\infty} \lim_{k \rightarrow +\infty} \langle \tau_{-s_{m_l}} \tau_{s_{m_k}} T, \tau_{-x} \check{\varphi} \rangle, \\ &= \langle T, \tau_{-x} \check{\varphi} \rangle, \\ &= (T * \varphi)(x), \end{aligned}$$

hence by proposition 1, $T * \varphi \in \mathcal{C}_{aa}$, $\forall \varphi \in \mathcal{D}$, i.e. $T \in \mathcal{B}'_{aa}$. \square

W. Stepanov introduced the class of almost periodic functions for which only local integrability in the sense of Lebesgue is required, this class was extended to the so called Stepanov almost automorphic functions, which is a more general concept than Bochner almost automorphy. We recall this definition, see [3].

Definition 3. A function $f \in L_{loc}^p$, $1 \leq p < \infty$, is said S^p -almost automorphic function ($S^p - a.a.$) if for every sequence $(s_m)_{m \in \mathbb{N}} \subset \mathbb{R}$, there is a subsequence $(s_{m_k})_{k \in \mathbb{N}}$ and a function $g \in L_{loc}^p$ such that

$$(5.2) \quad \lim_{k \rightarrow +\infty} \left(\int_0^1 |f(x + s_{m_k} + t) - g(x + t)|^p dt \right)^{\frac{1}{p}} = 0, \text{ for every } x \in \mathbb{R},$$

and

$$(5.3) \quad \lim_{k \rightarrow +\infty} \left(\int_0^1 |g(x - s_{m_k} + t) - f(x + t)|^p dt \right)^{\frac{1}{p}} = 0, \text{ for every } x \in \mathbb{R}.$$

It is well known that if $1 \leq p \leq q < \infty$, every $S^q - a.a.$ function is an $S^p - a.a.$ function, so the space of $S^1 - a.a.$ functions contains all the $S^p - a.a.$ functions.

Proposition 10. If f is S^1 -almost automorphic, then the associated distribution T_f is almost automorphic.

Proof. Recall that if $g \in L_{loc}^1$, $\langle T_g, \varphi \rangle := \int_{\mathbb{R}} g(t) \varphi(t) dt, \forall \varphi \in \mathcal{D}$. Let f be an S^1 -almost automorphic, then for every sequence $(s_m)_{m \in \mathbb{N}} \subset \mathbb{R}$, there exist a subsequence $(s_{m_k})_{k \in \mathbb{N}}$ and a function $g \in L_{loc}^1$, such that (5.2) and (5.3) are satisfied. Let T_g be the associated distribution to g and let us show that

$$T_g = \lim_{k \rightarrow +\infty} \tau_{s_{m_k}} T_f \text{ and } \lim_{k \rightarrow +\infty} \tau_{-s_{m_k}} T_g = T_f \text{ exist in } \mathcal{D}'.$$

Indeed, for $\varphi \in \mathcal{D}$ with $\text{supp} \varphi \subset [a, b]$ we can assume that a and b are integers, so

$$\begin{aligned} |\langle \tau_{s_{m_k}} T_f, \varphi \rangle - \langle T_g, \varphi \rangle| &= \left| \int_a^b (f(s_{m_k} + t) - g(t)) \varphi(t) dt \right| \\ &\leq \sup_{x \in \mathbb{R}} |\varphi(x)| \sum_{i=a}^{b-1} \int_i^{i+1} |f(s_{m_k} + t) - g(t)| dt, \\ &\leq \sup_{x \in \mathbb{R}} |\varphi(x)| \sum_{i=a}^{b-1} \int_0^1 |f(i + s_{m_k} + t) - g(i + t)| dt, \end{aligned}$$

consequently, $\lim_{k \rightarrow \infty} |\langle \tau_{s_{m_k}} T_f, \varphi \rangle - \langle T_g, \varphi \rangle| = 0$. In the same way, we obtain that

$$|\langle \tau_{-s_{m_k}} T_g, \varphi \rangle - \langle T_f, \varphi \rangle| \xrightarrow{k \rightarrow +\infty} 0,$$

hence the conclusion. \square

6. APPLICATIONS TO DIFFERENCE-DIFFERENTIAL EQUATIONS

We consider first linear difference-differential operators

$$(6.1) \quad L_h = \sum_{i=0}^p \sum_{j=0}^q a_{ij} \frac{d^i}{dx^i} \tau_{h_j},$$

where $(a_{ij})_{i \leq p, j \leq q}$ are complex numbers and $h = (h_j)_{j \leq q} \subset \mathbb{R}^q$.

Remark 6. *It is easy to see that $L_h T \in \mathcal{B}'_{aa}, \forall T \in \mathcal{B}'_{aa}$.*

Let $p \in \mathbb{Z}_+$ and denote by \mathcal{C}_{ub}^p the space of functions φ of class \mathcal{C}^p such that for every $j \leq p$, $\varphi^{(j)}$ is uniformly continuous and bounded on \mathbb{R} .

The following result is an extension of theorem 4 of [2].

Theorem 3. *If every solution $f \in \mathcal{C}_{ub}^p$ of the homogeneous equation*

$$(6.2) \quad L_h f = 0$$

is an almost automorphic function, then any solution $T \in \mathcal{D}'_{L^\infty}$ of the inhomogeneous equation

$$(6.3) \quad L_h T = S \in \mathcal{B}'_{aa}$$

is an almost automorphic distribution.

Proof. Let $T \in \mathcal{D}'_{L^\infty}$ be a solution of equation (6.3), then for every $\varphi \in \mathcal{D}$, we have

$$L_h (T * \varphi) = S * \varphi.$$

On the other hand, $\forall j \in \mathbb{Z}_+$, $(T * \varphi)^{(j)} = T * \varphi^{(j)} \in \mathcal{D}_{L^\infty}$, it is clear that $T * \varphi^{(j)}$ is a uniformly continuous function on \mathbb{R} , so $T * \varphi \in \mathcal{C}_{ub}^\infty$ and it is a solution of equation (6.3) with a second member $S * \varphi \in \mathcal{C}_{aa}$ instead of S , this together with the assumption on the homogeneous equation (6.2) imply that theorem 4 of [2] is applicable, consequently $T * \varphi \in \mathcal{C}_{aa}$, $\forall \varphi \in \mathcal{D}$, which gives $T \in \mathcal{B}'_{aa}$. \square

The following consequences of the theorem recapture the Bohl-Bohr result and the invariance by translation of the space of almost automorphic distributions.

Corollary 3. (1) *If $T \in \mathcal{D}'_{L^\infty}$ and $\frac{dT}{dx} \in \mathcal{B}'_{aa}$, then $T \in \mathcal{B}'_{aa}$.*

(2) If $T \in \mathcal{B}'_{aa}$, then $\tau_h T \in \mathcal{B}'_{aa}, \forall h \in \mathbb{R}$.

A particular case of the operators (6.1) are linear ordinary differential operators $L = \sum_{i=0}^p a_i \frac{d^i}{dx^i}$, these operators can be tackled in the more general situation of systems

$$(6.4) \quad U' = AU + S,$$

where $A = (a_{ij})_{1 \leq i, j \leq p}$ is a given square-matrix of complex numbers, also the vector distribution $S = (S_i)_{1 \leq i \leq p} \in (\mathcal{D}')^p$, and $U = (U_i)_{1 \leq i \leq p}$ is the unknown vector distribution.

Theorem 4. *If all S_i , $1 \leq i \leq p$ are almost automorphic distributions and the matrix A has no eigenvalues with real part zero, then the equation (6.4) admits a unique solution $U \in (\mathcal{D}'_{L^\infty})^p$ which is, in fact, an almost automorphic vector distribution.*

Proof. Consider the equation (6.4) and let $\varphi \in \mathcal{D}$, then

$$(U * \varphi)' = A(U * \varphi) + (S * \varphi),$$

where $U * \varphi = (U_i * \varphi)_{1 \leq i \leq p}$ and $(S * \varphi) = (S_i * \varphi)_{1 \leq i \leq p}$, which gives the following system of equations

$$v' = Av + g$$

with $g = S * \varphi \in (\mathcal{C}_{aa})^p$ and $v = U * \varphi \in (\mathcal{C}^\infty)^p$, consequently we apply theorem 2 of [7] to obtain that there exists a unique $v \in (\mathcal{C}_{aa})^p$, so $U_i * \varphi \in \mathcal{C}_{aa}, \forall \varphi \in \mathcal{D}, \forall i = 1, \dots, p$, i.e. $U \in (\mathcal{B}'_{aa})^p$. \square

We conclude with the following consequence of the last theorem.

Corollary 4. *If the polynomial $\sum_{i=0}^p a_i \lambda^i$ has no roots with real part zero, then any solution $T \in \mathcal{D}_{L^\infty}$ of the inhomogeneous equation $LT = S \in \mathcal{B}'_{aa}$ is an almost automorphic distribution.*

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